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1979 J. Phys. A: Math. Gen. 12 1781

(http://iopscience.iop.org/0305-4470/12/10/024)

J. Phys. A: Math. Gen., Vol. 12, No. 10, 1979. Printed in Great Britain

Equations of motion in linearised gravity: IV External fields

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Received 29 January 1979

Abstract. This paper is a sequel to recent papers on equations of motion in linearised gravity in which a new approach was adopted to study the motion of the sources of some Robinson–Trautman fields. We consider here the introduction of an external field to drive the source. We demonstrate this for the Levi-Civita fields of both a charged and uncharged uniformly accelerating mass.

1. Introduction

In a recent series of papers (Hogan and Imaeda 1979a, b, c) a new approach to studying the motion of the sources of some Robinson–Trautman (1962) fields in linearised gravity was described. This work was concerned with the self-fields of the particle-like sources. Under certain conditions it was found that the source (whether charged or uncharged) could perform uniform acceleration. In this case the field obtained was the linearised Levi-Civita (1918) solution of the vacuum (Einstein or Einstein–Maxwell) field equations. This solution has been studied by Kinnersley and Walker (1970) who were the first to point out that it possessed a curious type of nodal singularity. They conjectured that this was due to the absence of an external field to drive the particle. Recently Ernst (1976), using his complex potential framework, has shown how one might modify the charged Levi-Civita solution to introduce a constant electric field which, he has further shown, gives rise to the acceleration of the source 'for sufficiently small values of the acceleration'. He has also (Ernst 1978) suggested a procedure for introducing an external gravitational field to account for the constant acceleration of the uncharged Levi-Civita source.

In this paper we study the problem of introducing an external field to account for the acceleration of the linearised Levi-Civita solutions. Our approach is a natural extension of recent work (Hogan and Imaeda 1979a, b) dealing with the self-fields of the sources. We obtain agreement with Ernst (1976) for the case of a charged source, under stringent conditions of approximation. Our result in the uncharged case runs contrary to a conjecture contained in Ernst (1978). We expect that the general procedure followed in this paper could be applied to the introduction of external fields to drive the sources of other Robinson–Trautman solutions, and work is continuing along these lines.

The outline of the paper is as follows: in § 2 we describe the linearised Levi-Civita solution for a charged source and draw attention to the nodal singularity referred to above, in a manner given by Robinson and Robinson (1972). As an indication of our procedure, we consider in § 3 a simple candidate for the driving field of a uniformly

0305-4470/79/101781+13$01.00

accelerating mass, suggested by Ernst (1978). We find this candidate unacceptable. The interaction of an external electromagnetic field with the field of a charged mass is contained in the nonlinear electromagnetic energy–momentum tensor. By solving the linearised Einstein–Maxwell vacuum field equations, with the electromagnetic field composed of the self-field of the charge and a constant electric field, we demonstrate in § 4 how one finds that the nodal singularity is removed. This is followed by a discussion in § 5.

2. The nodal singularity

The linearised field of a uniformly accelerating charged mass is described in Robinson–Trautman form by the line element

$$ds^2 = 2r^2P^{-2}\,d\zeta\,d\bar{\zeta} - 2\,d\sigma\,dr - h\,d\sigma^2 \tag{2.1}$$

where

$$P = -2k_0(1-\xi_0)^{-1}[1 - ma\xi_0\ln(1-\xi_0^2) + \tfrac{3}{2}e^2a^2(1-\xi_0^2)] + O_2 \tag{2.2a}$$

$$h = K - 2Hr - 2(m + 2e^2a\xi_0)r^{-1} + e^2r^{-2} + O_2 \tag{2.2b}$$

$$K = 1 + 6ma\xi_0 + 6e^2a^2\xi_0^2 + O_2 \tag{2.2c}$$

$$H = a\xi_0 + ma^2\{2\xi_0^2 - (1-\xi_0^2)\ln(1-\xi_0^2)\} - 3e^2a^3\xi_0(1-\xi_0^2) + O_2. \tag{2.2d}$$

On a suitable null tetrad in Newman–Penrose (1962) notation, the only nonvanishing components of the linearised Weyl and Maxwell tensors are

$$\psi_2 = -(m + 2e^2a\xi_0)r^{-3} + e^2r^{-4} + O_2 \tag{2.3a}$$

$$\Phi_1 = -e(2r^2)^{-1} + O_2 \tag{2.3b}$$

respectively. Here the constants m, e and a are the mass, charge and acceleration of the source $r = 0$ when viewed in the background Minkowskian space–time, with line element given by equations (2.1) and (2.2) with $m = 0$. They are small in the sense that we consider the dimensionless (using units for which $c = G = 1$) constants ma, e^2a^2, both small of first order, writing $ma = O_1$, $e^2a^2 = O_1$. Also

$$\xi_0 = (\tfrac{1}{2}\zeta\bar{\zeta} - k_0^2)/(\tfrac{1}{2}\zeta\bar{\zeta} + k_0^2) \qquad (k_0 = -\exp(-a\sigma)). \tag{2.4}$$

For a detailed description of the construction of the solution (2.1) and (2.2) the reader is referred to Hogan and Imaeda (1979a, b). We show in Appendix A how one transforms equations (2.1) and (2.2) into a more familiar form of the linearised charged Levi-Civita solution. We point out here that in the background Minkowskian space–time of equation (2.1), putting $m = 0$ in equation (2.2), the world-line $r = 0$ is time-like with constant acceleration a and with σ as proper time along it. The future null cones with vertices on $r = 0$ have equations σ = constant; r is the affine parameter along the generators of the null cones and these generators are labelled (on each null cone) by the polar coordinates θ, ϕ which are related to the complex coordinate ζ by $\zeta = \sqrt{2}\exp(i\phi)\tan(\theta/2)$. We notice that, although some of the metric tensor components (2.2) are singular when $\xi_0 = \pm 1$, which corresponds to a pair of generators $\theta = 0$, π on each future null cone σ = constant in the background space–time, the fields given by equation (2.3) are only singular on $r = 0$, as one would expect of the fields of a simple pole particle.

We now examine the 2-surfaces $\sigma = \text{const.}$, $r = \text{const.}$ in the space–time described by equation (2.1). These have line element

$$dl^2 = 2r^2P^{-2}\,d\zeta\,d\bar{\zeta}, \tag{2.5}$$

with P given by equation (2.2). One can show that K in equation (2.2) is the Gaussian curvature of this 2-surface. We will now briefly summarise how one looks for nodal or conical singularities on this 2-surface in the manner suggested by Robinson and Robinson (1972).

The 2-surface in question is axially symmetric and one can find a coordinate transformation (given below) from ζ, $\bar{\zeta}$ to ξ, η such that (2.5) takes the form

$$dl^2 = r^2(f^{-1}\,d\xi^2 + f\,d\eta^2) \tag{2.6}$$

where $f = f(\xi)$, $f(\xi) > 0$ for $\xi_1 < \xi < \xi_2$ and $f(\xi_1) = 0 = f(\xi_2)$. One assumes that $f \in C^2[\xi_1, \xi_2]$ and one can show that

$$\int_{\xi_1}^{\xi_2} \xi K\,d\xi = -\tfrac{1}{2}(f'(\xi_1) + f'(\xi_2)) \tag{2.7}$$

where K is the Gaussian curvature of the 2-surface and the prime denotes differentiation with respect to ξ. Now consider the curve with parametric equations

$$x(\xi) = \int \sqrt{f^{-1}\left[1 - \left(\frac{f'}{2\mu}\right)^2\right]}\,d\xi, \qquad y(\xi) = \sqrt{f}/\mu \tag{2.8}$$

in Euclidean 2-space where μ is a constant such that 2μ is an upper bound for $|f'(\xi)|$ for $\xi \in [\xi_1, \xi_2]$. We note that $y = 0$ when $\xi = \xi_1$ or ξ_2 and the gradient of the tangent to the curve is given by

$$dy/dx = \pm[(2\mu/f')^2 - 1]^{-1/2}. \tag{2.9}$$

If this curve is rotated about the x axis through an angle $\phi = \mu\eta$ it generates a 2-surface with line element given by (2.6) with $r = 1$. If $0 \leq \phi < 2\pi$ then, assuming $\mu > 0$, η lies in the range $0 \leq \eta < 2\pi\mu^{-1}$. A nodal or conical singularity will appear at the ends corresponding to $\xi = \xi_1$ or ξ_2 of the 2-surface, where $y = 0$, unless the tangent to the curve (2.8) is inclined at 90° to the x axis where $\xi = \xi_1$ or ξ_2. By equation (2.9) this will be achieved at $\xi = \xi_1$ if† $\mu = \frac{1}{2}f'(\xi_1)$, i.e. by a choice of the range of η. If this choice is made, then the node at $\xi = \xi_2$ will be absent if, using equation (2.7),

$$\int_{\xi_1}^{\xi_2} \xi K\,d\xi = 0. \tag{2.10}$$

We may write equation (2.5) in the form of equation (2.6), with

$$f(\xi) = (1 - \xi^2)(1 + 2ma\xi) - e^2a^2(3 + \xi^4) \tag{2.11}$$

using the transformation

$$\eta = -i/2\ln(\zeta\bar{\zeta}^{-1}) \tag{2.12a}$$

$$\xi = \xi_0 + ma(1 - \xi_0^2)(1 - \ln(1 - \xi_0^2)) - e^2a^2\xi_0(3 - \xi_0^2) \tag{2.12b}$$

† This condition may not in general be compatible with 2μ being an upper bound of $|f'(\xi)|$. In our subsequent applications $f = 1 - \xi^2 + O_1$, $\xi_1 = -1 + O_1$, $\xi_2 = +1 + O_1$ and if the O_1 term is sufficiently small there will be no incompatibility.

with ξ_0 given by equation (2.4). We note that we no longer have $\zeta = \sqrt{2}\exp(\mathrm{i}\phi)\tan(\theta/2)$, as with the case in the background Minkowskian space–time, but we may assume $\zeta = \sqrt{2}\exp(\mathrm{i}\mu^{-1}\phi)\tan(\theta/2)$, so that as in the previous paragraph $\phi = \mu\eta$. We shall find that $\mu = 1 + O_1$. If we put $e = 0$ in (2.11) we recover the value of $f(\xi)$ obtained by Robinson and Robinson (1972). We calculate from equation (2.11) that $f(\xi) = 0$ when $\xi = \xi_1 = -1 + 2e^2a^2 + O_2$ and when $\xi = \xi_2 = 1 - 2e^2a^2 + O_2$. One easily sees that $f(\xi) > 0$ for $\xi_1 < \xi < \xi_2$ when ma and e^2a^2 are sufficiently small. Clearly $f(\xi) \in C^2[\xi_1, \xi_2]$. The node at $\xi = \xi_1$ is removed by choosing $\mu = \frac{1}{2}f'(\xi_1) = 1 - 2ma + O_2$. Thus $\eta = (1 + 2ma)\phi + O_2$. The quantity K in equation (2.10) is given in (2.2). We may replace ξ_0 by ξ in this expression with the residual terms going into the O_2 error. Calculating the integral in equation (2.10) we obtain

$$\int_{\xi_1}^{\xi_2} \xi K \, \mathrm{d}\xi = 4ma + O_2. \tag{2.13}$$

Hence, so long as $a \neq 0$, and whether or not $e = 0$, the 2-surface possesses a nodal or conical singularity at the end corresponding to $\xi = \xi_2$. This curiosity was first observed by Kinnersley and Walker (1970) and it is the purpose of the rest of this paper to examine their conjecture: if an external field is introduced to drive the source then an additional O_1 term will apear on the right-hand side of equation (2.13) to cancel the existing one. This will provide us with an equation of motion for the source *and* the elimination of the nodal singularity, modulo an O_2 error.

3. The uncharged source

In this section we are interested in modifying the self-field of an uncharged uniformly accelerating mass by the addition of an external field. The self-field in question is thus obtained from equations (2.1) and (2.2) by putting $e = 0$.

Let $X^i = (x, y, z, t)$ be rectangular Cartesian coordinates and time in the background Minkowskian space–time. The history of the source (the time-like world-line $r = 0$) in this background space–time is given by $X^i = x^i(\sigma) = a^{-1}(\cosh a\sigma)\delta^i_3 + a^{-1}(\sinh a\sigma)\delta^i_4$, and the coordinates X^i are related to the coordinates $(\zeta, \bar{\zeta}, r, \sigma)$ in the Minkowskian background by (cf Hogan and Imaeda 1979a)

$$X^i = x^i(\sigma) + rP_0^{-1}\zeta^i \tag{3.1a}$$

$$P_0 = \exp(a\sigma)[\tfrac{1}{2}\zeta\bar{\zeta} + \exp(-2a\sigma)] \tag{3.1b}$$

$$\zeta^i = (1/\sqrt{2})(\zeta + \bar{\zeta})\delta^i_1 + (1/\mathrm{i}\sqrt{2})(\zeta - \bar{\zeta})\delta^i_2 + (1 - \tfrac{1}{2}\zeta\bar{\zeta})\delta^i_3 + (1 + \tfrac{1}{2}\zeta\bar{\zeta})\delta^i_4. \tag{3.1c}$$

Here also, since we are in the background space–time, we have $\zeta = \sqrt{2}\exp(\mathrm{i}\phi)\tan(\theta/2)$.

The field described by equations (2.1) and (2.2) is axially symmetric and so it is reasonable to look for an external field which is axially symmetric. We therefore consider an external field, described by a tensor $\bar{\gamma}_{ij}$, of the static Weyl form, given by (cf Synge 1966, p 312, in linearised form)

$$\bar{\gamma}_{ij}\, \mathrm{d}\bar{X}^i\, \mathrm{d}\bar{X}^j = 2(\nu - \lambda)(\mathrm{d}\rho^2 + \mathrm{d}z^2) - 2\lambda\rho^2\, \mathrm{d}\phi^2 - 2\lambda\, \mathrm{d}t^2 \tag{3.2}$$

where

$$\bar{X}^i = (\rho, \phi, z, t), \qquad \nu = \nu(\rho, z), \qquad \lambda = \lambda(\rho, z),$$

$$x = \rho\cos\phi, \qquad \text{and} \qquad y = \rho\sin\phi.$$

The O_1 quantities ν, λ satisfy the field equations

$$\partial^2\lambda/\partial\rho^2+(1/\rho)(\partial\lambda/\partial\rho)+\partial^2\lambda/\partial z^2=0 \tag{3.3a}$$

$$\nu=\int r\{(\partial\lambda/\partial\rho)^2-(\partial\lambda/\partial z)^2\}\,\mathrm{d}r+2r(\partial\lambda/\partial\rho)(\partial\lambda/\partial z)\,\mathrm{d}z=O_2. \tag{3.3b}$$

As solution of equation (3.3a) we choose

$$\lambda=kz \qquad k=\text{constant}. \tag{3.4}$$

We are here taking the linearised form of the external field suggested by Ernst (1978). With this choice we have, from equation (3.2),

$$\bar{\gamma}_{ij}\,\mathrm{d}\bar{X}^i\,\mathrm{d}\bar{X}^j=-2\lambda(\bar{\eta}_{ij}\,\mathrm{d}\bar{X}^i\,\mathrm{d}\bar{X}^j+2\,\mathrm{d}t^2)+O_2. \tag{3.5}$$

We are henceforth specifically interested in the contribution made by equation (3.5) to the 2-surface $\sigma=$ constant, $r=$ constant. We may simplify our calculations, without loss of generality, if we take this 2-surface to be $\sigma=0$, $r=r_0$(constant). Under these assumptions, and transforming from $\bar{X}^i\to X^i\to(\zeta,\bar{\zeta},r,\sigma)$ in equation (3.5), we easily find that on the 2-surface $\sigma=0$, $r=r_0$

$$\bar{\gamma}_{ij}\,\mathrm{d}\bar{X}^i\,\mathrm{d}\bar{X}^j=-4\lambda r_0^2P_0^{-2}\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}+O_2 \tag{3.6}$$

where now, by equation (3.1),

$$P_0=\tfrac{1}{2}\zeta\bar{\zeta}+1 \tag{3.7a}$$

$$\lambda=k\{a^{-1}+r_0[(1-\tfrac{1}{2}\zeta\bar{\zeta})/(1+\tfrac{1}{2}\zeta\bar{\zeta})]\}=O_1. \tag{3.7b}$$

Adding equation (3.6) to equation (2.5) we find that the line element of the 2-surface $\sigma=0$, $r=r_0$ for the combined self-field plus external field is

$$\mathrm{d}\hat{l}^2=2r_0^2P_0^{-2}(1-2\hat{Q})\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} \tag{3.8a}$$

$$\hat{Q}=-ma\xi_0\ln(1-\xi_0^2)+\lambda+O_2 \tag{3.8b}$$

where P_0 is given by equation (3.7a), λ by equation (3.7b) and ξ_0 by equation (2.4) with $\sigma=0$. We now transform equation (3.8a) into the form (2.6) with

$$f(\xi)=(1-\xi^2)[1+2ma\xi-2ka^{-1}(1-r_0a\xi)] \tag{3.9}$$

by the transformation

$$\eta=(-\mathrm{i}/2)\ln\zeta\bar{\zeta}^{-1} \tag{3.10a}$$

$$\xi=\xi_0+ma(1-\xi_0^2)[1-\ln(1-\xi_0^2)]-2ka^{-1}(1-\tfrac{1}{2}r_0a\xi_0^2). \tag{3.10b}$$

We see from equation (3.9) that $f(\xi)=0$ for $\xi=\pm1$. The Gaussian curvature of equation (3.8a) is given by $\hat{K}=-\tfrac{1}{2}f''$. One easily finds that now $\mu=1-2ma-2ka^{-1}-2kr_0$ for the removal of the node on equation (3.8a) at $\xi=-1$, while the node at $\xi=+1$ will disappear in the linear approximation if

$$\int_{-1}^{+1}\hat{K}\xi\,\mathrm{d}\xi=4(ma+kr_0)+O_2=O_2 \tag{3.11}$$

that is, if

$$ma=-kr_0+O_2. \tag{3.12}$$

Since m, a, k are constants it is impossible to satisfy equation (3.12) for more than one value of r_0. Hence we rule the external field given by (3.5) as unacceptable. This result is a counter-example, albeit in the linear approximation, to a recent conjecture by Ernst (1978).

It would appear that one should examine a two-body problem to obtain a solution which is acceptable on all grounds.

4. The charged source

We now consider an external electromagnetic field to provide the uniform acceleration of the charged particle whose self-field is described by equations (2.1) and (2.2). The obvious external field to try is a constant electric field E in the z direction. In the coordinates X^i this means that the only nonvanishing components of the electromagnetic tensor for the external field are $F^{\text{ext}}_{34} = -F^{\text{ext}}_{43} = E$. We shall choose the dimensionless quantity $a^{-2}E^2 = O_1$. The self-electromagnetic field F^{self}_{ij} of the charge is the Liénard–Wiechert field (Synge 1970)

$$F^{\text{self}}_{ij} = er^{-1}(U_i k_j - U_j k_i) \tag{4.1a}$$

$$U^i = \mu^i + B\lambda^i \tag{4.1b}$$

$$B = (1 + r\mu^j k_j)r^{-1} \tag{4.1c}$$

where $k^i = P_0^{-1}\zeta^i$, with P_0 and ζ^i given by equation (3.1), $\lambda^i = \sinh a\sigma\delta^i_3 + \cosh a\sigma\delta^i_4$ is the 4-velocity of the source and $\mu^i = a\cosh a\sigma\delta^i_3 + a\sinh a\sigma\delta^i_4$ is the 4-acceleration of the source in the background Minkowskian space–time. The complete electromagnetic field is

$$F_{ij} = F^{\text{self}}_{ij} + F^{\text{ext}}_{ij}. \tag{4.2}$$

The electromagnetic energy–momentum tensor is obtained from

$$\begin{aligned} E_{ij} &= \eta^{ab}F_{ai}F_{bj} - \tfrac{1}{4}\eta_{ij}F_{ab}F^{ab} + O_2 \\ &= E^{\text{self}}_{ij} + E^{\text{int}}_{ij} + E^{\text{ext}}_{ij} + O_2 \end{aligned} \tag{4.3}$$

the second equality arising from the substitution of equation (4.2) into equation (4.3). The vacuum Einstein–Maxwell linearised field equations read (in units for which $4\pi = 1$)

$$R_{ij} = -2(E^{\text{self}}_{ij} + E^{\text{int}}_{ij} + E^{\text{ext}}_{ij}) + O_2 \tag{4.4}$$

where R_{ij} is the linearised Ricci tensor for a metric of the form

$$g_{ij} = \eta_{ij} + \gamma_{ij} \tag{4.5}$$

where $\gamma_{ij} = O_1$. We may write

$$\gamma_{ij} = \gamma^{\text{self}}_{ij} + \gamma^{\text{int}}_{ij} + \gamma^{\text{ext}}_{ij} \tag{4.6}$$

with the self-field $\gamma^{\text{self}}_{ij}$ satisfying

$$R_{ij}(\gamma^{\text{self}}) = -2E^{\text{self}}_{ij} \tag{4.7}$$

and the interaction field γ^{int}_{ij} and pure external field γ^{ext}_{ij} satisfying

$$R_{ij}(\gamma^{\text{int}}) = -2E^{\text{int}}_{ij} \qquad R_{ij}(\gamma^{\text{ext}}) = -2E^{\text{ext}}_{ij} \tag{4.8}$$

respectively. We already have found $\gamma_{ij}^{\text{self}}$. It is given in terms of the coordinates $(\zeta, \bar{\zeta}, r, \sigma)$ by equations (2.1) and (2.2). Our task now is to solve equation (4.8) for γ_{ij}^{int} and γ_{ij}^{ext}. The pure external field is the simplest to deal with and we shall discuss it briefly first.

In coordinates $X^i = (x, y, z, t)$ we find that the nonvanishing components of E_{ij}^{ext} are

$$E_{11}^{\text{ext}} = E_{22}^{\text{ext}} = -E_{33}^{\text{ext}} = E_{44}^{\text{ext}} = \tfrac{1}{2}E^2 = O_1. \tag{4.9}$$

Transforming to coordinates $\bar{X}^i = (\rho, \phi, z, t)$ in the manner of the previous section we find $\bar{E}_{11}^{\text{ext}} = \rho^{-2}\bar{E}_{22}^{\text{ext}} = -\bar{E}_{33}^{\text{ext}} = \bar{E}_{44}^{\text{ext}} = \tfrac{1}{2}E^2$. From these expressions it is easy to verify that $\bar{E}_2^2 + \bar{E}_4^4 = 0$ and so (cf Synge 1966) we may take $\bar{\gamma}_{ij}^{\text{ext}}$ in the static Weyl form as in equation (3.2), i.e.

$$\bar{\gamma}_{ij}^{\text{ext}}\, \mathrm{d}\bar{X}^i\, \mathrm{d}\bar{X}^j = 2(\nu - \lambda)(\mathrm{d}\rho^2 + \mathrm{d}z^2) - 2\lambda\rho^2\, \mathrm{d}\phi^2 - 2\lambda\, \mathrm{d}t^2 \tag{4.10}$$

with $\nu(\rho, z) = O_1$ and $\lambda(\rho, z) = O_1$. The linearised field equations, the second of equations (4.8), are now

$$\partial^2\nu/\partial z^2 + \partial^2\nu/\partial\rho^2 = \partial^2\lambda/\partial\rho^2 + \partial^2\lambda/\partial z^2 + (1/\rho)\partial\lambda/\partial\rho \tag{4.11a}$$

$$(1/\rho)\partial\nu/\partial\rho = E^2 \tag{4.11b}$$

$$\partial\nu/\partial z = 0 \tag{4.11c}$$

$$\partial^2\lambda/\partial\rho^2 + (1/\rho)\,\partial\lambda/\partial\rho + \partial^2\lambda/\partial z^2 = E^2. \tag{4.11d}$$

Making the reasonable assumption that λ and ν are both functions of ρ only, we find

$$\lambda = \tfrac{1}{4}\rho^2E^2 + a_0\ln\rho + a_1 \tag{4.12a}$$

$$\nu = \tfrac{1}{2}\rho^2E^2 + a_2 \tag{4.12b}$$

where a_0, a_1, a_2 are constants of integration. On substituting these into the Weyl tensor we find that it is singular on $\rho = 0$ unless $a_0 = 0$. $\rho = 0$ is the z axis so this would constitute a 'wire singularity' in the Weyl tensor if $a_0 \neq 0$. The constants a_1, a_2 do not contribute to the field (Weyl tensor) and can thus be removed by a trivial gauge transformation. The resulting Weyl tensor components are given in Appendix B. Hence equation (4.10) becomes

$$\bar{\gamma}_{ij}^{\text{ext}}\, \mathrm{d}\bar{X}^i\, \mathrm{d}\bar{X}^j = 2\lambda[\bar{\eta}_{ij}\, \mathrm{d}\bar{X}^i\, \mathrm{d}\bar{X}^j - 2\rho^2\, \mathrm{d}\phi^2] \tag{4.13a}$$

$$\lambda = \tfrac{1}{4}\rho^2E^2 = O_1. \tag{4.13b}$$

On the 2-surface $\sigma = 0$, $r = r_0$ with $\rho = r_0\sin\theta$, $\zeta = \sqrt{2}\exp(\mathrm{i}\phi)\tan(\theta/2) + O_1$, and passing from $\bar{X}^i$ to X^i to $(\zeta, \bar{\zeta}, r, \sigma)$ via equation (3.1), we find that equation (4.13a) gives the contribution

$$\mathrm{d}l_1^2 = 2\lambda[2r_0^2P_0^{-2}\, \mathrm{d}\zeta\, \mathrm{d}\bar{\zeta} - 2\rho^2\, \mathrm{d}\phi^2]. \tag{4.14}$$

We now move to the interaction field. We shall find that components of the interaction energy–momentum tensor depend upon the coordinate σ as well as the other coordinates, on account of equation (4.1). Because of the complexity this would otherwise bring to the problem, we shall henceforth make the 'slow-motion assumption'

$$a\sigma = O_1. \tag{4.15}$$

This means that the history of the particle in the background space–time does not differ greatly from a geodesic (see figure 1). Also if $f(\sigma) = O_1$ then we may write $f(\sigma) =$

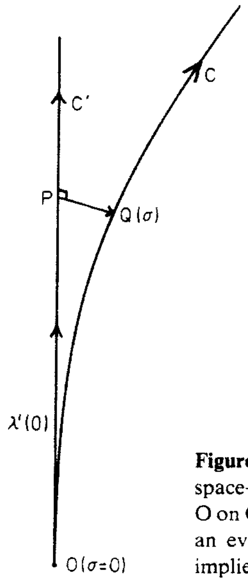

Figure 1. C is the time-like world line of the source in the background space–time. Q is an event on C a parametric distance σ from the event O on C. C′ is the time-like geodesic through O with tangent $\lambda^{\prime}(0)$. P is an event on C′ a parametric distance σ from O. Equation (4.15) implies $PQ/OQ = \frac{1}{2}a\sigma + O_2$.

$f(0) + O_2$. We shall do this for the components E_{ij}^{int}. Hence we shall only be solving the first of equations (4.8) for γ_{ij}^{int} in the region R between the future null cones $\sigma = 0$ and $\sigma = \text{constant}$, where $a\sigma = O_1$ (see figure 2). The field can be considered independent of σ in this region.

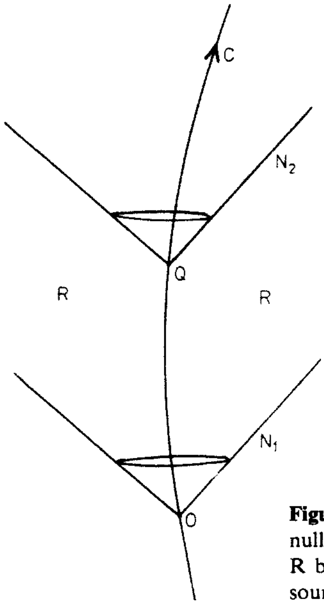

Figure 2. N_1 is the future null cone $\sigma = 0$. N_2 is the future null cone $\sigma = \text{const}$. We solve for γ_{ij}^{int} at events in the region R between N_1 and N_2 and in the neighbourhood of the source world line C, on account of equation (4.18).

Calculating E_{ij}^{int} and then $\bar{E}_{ij}^{\text{int}}$, and making the assumption (4.15), we find the nonvanishing components

$$\bar{E}_{11}^{\text{int}} = \bar{E}_{22}^{\text{int}} = -\bar{E}_{33}^{\text{int}} = \bar{E}_{44}^{\text{int}} = -eE\rho^{-1} \sin\theta(a - az\rho^{-1} \sin\theta \cos\theta) + O_2 \qquad (4.16a)$$

$$\bar{E}_{13}^{\text{int}} = -eEaz\rho^{-2}\sin^3\theta + O_2 \tag{4.16b}$$

$$\bar{E}_{14}^{\text{int}} = eEa\rho^{-1}\sin^2\theta + O_2 \tag{4.16c}$$

where, from equation (3.1),

$$\rho = r\sin\theta + O_1, \qquad z = a^{-1} + r\cos\theta + O_1. \tag{4.17}$$

We now make a further approximation, namely that

$$ar = O_1. \tag{4.18}$$

This confines the range of r, the retarded distance from the source world line in the background space–time. Thus $az = 1 + O_1$ and we may neglect $\bar{E}_{14}^{\text{int}}$ in favour of $\bar{E}_{13}^{\text{int}}$. Also

$$\bar{E}_{11}^{\text{int}} = eEr^{-2}\cos\theta + O_2. \tag{4.19}$$

Once again we find that $\bar{E}_2^2 + \bar{E}_4^4 = 0$ and so we may take $\bar{\gamma}_{ij}^{\text{int}}$ in the static Weyl form just as in equation (4.10). We shall use $\hat{\nu}$, $\hat{\lambda}$ this time in place of ν, λ. If we make the change of variable (4.17), the field equations, the first of equations (4.8), read

$$r^2\cos\theta\,\partial^2\hat{\nu}/\partial r^2 + \sin\theta\,\partial\hat{\nu}/\partial\theta + \cos\theta\,\partial^2\hat{\nu}/\partial\theta^2 = 2eE + O_2 \tag{4.20a}$$

$$\partial\hat{\nu}/\partial\theta - r\cot\theta\,\partial\hat{\nu}/\partial r = 2eE\sin\theta + O_2 \tag{4.20b}$$

$$\partial^2\hat{\lambda}/\partial r^2 + (2/r)\,\partial\hat{\lambda}/\partial r + (1/r^2)\,\partial^2\hat{\lambda}/\partial\theta^2 + (\cot\theta/r^2)\,\partial\hat{\lambda}/\partial\theta = 2eEr^{-2}\cos\theta + O_2. \tag{4.20c}$$

These may be solved by a separation of variables, both for $\hat{\nu}(r,\theta)$ and $\hat{\lambda}(r,\theta)$, of the form $g(r) + h(\theta)$. The general solutions are

$$\hat{\lambda} = -eE\cos\theta + b_0\ln r\sin\theta - \tfrac{1}{2}b_1\ln[(1+\cos\theta)/(1-\cos\theta)] - b_2/r + b_3 \tag{4.21a}$$

$$\hat{\nu} = -2eE\cos\theta + c_0 \tag{4.21b}$$

where the b and c_0 are constants of integration. On substituting these expressions into the interaction linearised Weyl tensor we find that the field is singular when $\theta = 0$ or π (these are wire singularities) unless $b_0 = b_1 = 0$. The constants b_3 and c_0 can be removed by a gauge transformation. This leaves only b_2. The resulting components of the Weyl tensor are given in Appendix B, where it is further shown that the b_2 term would correspond to a superimposed linearised Schwarzschild field of mass b_2. The history of this source in the background Minkowskian space–time would be the time-like world line C' in figure 1. We therefore take $b_2 = 0$ and thus

$$\hat{\nu} = 2\hat{\lambda} = -2eE\cos\theta. \tag{4.22}$$

Hence $\bar{\gamma}_{ij}^{\text{int}}$ is given by

$$\bar{\gamma}_{ij}^{\text{int}}\,d\bar{X}^i\,d\bar{X}^j = 2\hat{\lambda}(\bar{\eta}_{ij}\,d\bar{X}^i\,d\bar{X}^j - 2\rho^2\,d\phi^2) \tag{4.23}$$

which is similar in form to equation (4.13) since in that case also $\nu = 2\lambda$. The contribution this makes to the line-element of the 2-surface $\sigma = 0$, $r = r_0$ is, in similar fashion to equation (4.14),

$$dl_2^2 = 2\hat{\lambda}(2r_0^2P_0^{-2}\,d\zeta\,d\bar{\zeta} - 2\rho^2\,d\phi^2) \tag{4.24}$$

where P_0 is given by equation (3.7a).

To obtain the line element on the 2-surface $\sigma = 0$, $r = r_0$ of the complete field given by equations (4.5) and (4.6), we must add together equation (2.5) with $r = r_0$, equations

(4.14) and (4.24). We arrive at

$$d\hat{l}^2 = 2r_0^2P_0^{-2}(1-2Q+2\Lambda)\,d\zeta\,d\bar{\zeta} - 4\Lambda\rho^2\,d\phi^2 \qquad (4.25a)$$

$$Q = -ma\xi_0 \ln(1-\xi_0^2) + \tfrac{3}{2}e^2a^2(1-\xi_0^2) \qquad (4.25b)$$

$$\Lambda = \lambda + \hat{\lambda} \qquad (4.25c)$$

with an O_2 error. We can write this in the form (2.6) with

$$f(\xi) = (1-\xi^2)(1-2eE\xi+2ma\xi) - e^2a^2(3+\xi^4) \qquad (4.26)$$

using the transformation

$$\eta = (-i/2)\ln \zeta\bar{\zeta}^{-1} = \phi + O_1 \qquad (4.27a)$$

$$\xi = \xi_0 + ma(1-\xi_0^2)(1-\ln(1-\xi_0^2)) - e^2a^2\xi_0(3-\xi_0^2). \qquad (4.27b)$$

We see from equation (4.26) that $f(\xi)=0$ when $\xi=\xi_1$ or ξ_2 with $\xi_1 = -1+2e^2a^2$ $\xi_2 = 1-2e^2a^2$. The node at $\xi=\xi_1$ is removed (cf § 2) by choosing $\mu = 1-2ma+2eE$ so that the O_1 term in equation (4.27) is $2ma\phi - 2eE\phi$. The Guassian curvature of the 2-surface (4.25a) is given by $\hat{K} = -\frac{1}{2}f''$ and one then has

$$\int_{\xi_1}^{\xi_2} \xi\hat{K}\,d\xi = 4(ma-eE)+O_2 \qquad (4.28)$$

and so the node at $\xi=\xi_2$ is removed in the linear approximation, provided

$$ma = eE + O_2. \qquad (4.29)$$

This is just the equation of motion one would expect to have. We have obtained here a result in agreement with Ernst (1976) but only under very stringent conditions of approximation. Comparing equation (4.28) with equation (2.13) we see that the unwanted node on the 2-surface $\sigma=0$, $r=r_0$ can be removed with a suitable choice of external field.

5. Discussion

One might presume that within our approximation scheme, and in the light of our result in § 4, a charged particle moving in an external electromagnetic field will, in general, move according to the Lorentz equation of motion. This, however, will have to be proved. The case of a neutral particle is somewhat more difficult. Clearly what is needed is a solution for a two-body system, within the above scheme. This would appear to be tractable and might also be of astrophysical significance. It is hoped that the present paper both complements our previous work and sheds some light on an approach to these problems.

Finally, it is interesting to note that when the problem of establishing the motion of extended sources is tackled in a systematic way (cf Hogan and McCrea 1974) one makes the assumption of slow-motion and one calculates the field only in the neighbourhood of the sources. Retarded potentials are expanded in terms of instantaneous potentials--an expansion which is only valid near the sources. Both of these conditions are reflected in our assumptions (4.15) and (4.18).

Acknowledgments

I wish to thank Drs J D McCrea and G M O'Brien and Mari Imaeda for their interest in this work and for many stimulating discussions. I am indebted to Dr McCrea for a critical reading of the manuscript.

Appendix A

The exact Levi-Civita (1918) solution of the vacuum Einstein–Maxwell field equations may be written in the form (cf Kinnersley and Walker 1970)

$$ds^2 = r^2(G^{-1}\,d\xi^2 + G\,d\eta^2) - 2ar^2\,d\sigma\,d\xi - 2\,dr\,d\sigma - C\,d\sigma^2 \qquad (A.1)$$

where

$$G = 1 - \xi^2 - 2ma\xi^3 - e^2a^2\xi^4 \qquad (A.2)$$

$$\begin{aligned} C = {} & 1 + 6ma\xi + 6e^2a^2\xi^2 - 2ar(\xi + 3ma\xi^2 - 2e^2a^2\xi^3) \\ & - a^2r^2(1 - \xi^2 - 2ma\xi^3 - e^2a^2\xi^4) - 2(m + 2e^2a\xi)r^{-1} + e^2r^{-2}. \end{aligned} \qquad (A.3)$$

The linearised form of this is

$$ds^2 = r^2G[(b\,d\xi - a\,d\sigma)^2 + d\eta^2] - 2\,dr\,d\sigma - h\,d\sigma^2 + O_2 \qquad (A.4)$$

where

$$b = (1-\xi^2)^{-1}[1 + (2ma\xi^3 + e^2a^2\xi^4)(1-\xi^2)^{-1}] \qquad (A.5)$$

$$h = 1 + 6ma\xi + 6e^2a^2\xi^2 - 2ar(\xi + 3ma\xi^2 - 2e^2a^2\xi^3) - 2(m + 2e^2a\xi)r^{-1} + e^2r^{-2}. \qquad (A.6)$$

If one makes the transformation

$$\begin{aligned} \eta &= (-i/2)\ln \zeta\bar{\zeta}^{-1} \\ \xi &= \xi_0 - ma\xi_0^2 - ma(1-\xi_0^2)\ln(1-\xi_0^2) - \tfrac{3}{2}e^2a^2\xi_0 \\ &\quad + e^2a^2\xi_0^3 + \tfrac{3}{4}e^2a^2(1-\xi_0^2)\ln[(1+\xi_0)/(1-\xi_0)] \end{aligned} \qquad (A.7)$$

with ξ_0 given by equation (2.4), one finds that equation (A.4) takes the form (2.1), with P and h given by

$$\begin{aligned} P &= -2k_0(1-\xi_0)^{-1}[1 - ma\xi_0\ln(1-\xi_0^2) + \tfrac{3}{2}e^2a^2(1-\xi_0^2) + \tfrac{3}{2}e^2a^2Q_1(\xi_0)] + O_2, \\ h &= K - 2Hr - 2(m + 2e^2a\xi_0)r^{-1} + e^2r^{-2} + O_2 \\ K &= 1 + 6ma\xi_0 + 6e^2a^2\xi_0^2 + O_2 \\ H &= a\xi_0 + ma[2\xi_0^2 - (1-\xi_0^2)\ln(1-\xi_0^2)] - 3e^2a^3\xi_0(1-\xi_0^2) + \tfrac{3}{2}e^2a^2(\partial/\partial\sigma)Q_1(\xi_0) + O_2. \end{aligned} \qquad (A.8)$$

This is in agreement with equation (2.2) except for the term involving $Q_1(\xi_0)$ in P and H. Here $Q_1(\xi_0)$ is the $l = 1$ Legendre function of the second kind. It has been shown in Hogan and Imaeda (1979a, equations (2.7), (2.8), 1979b equations (3.13)–(3.15)) that such a term may be removed by a gauge transformation provided its coefficient, in this case $\frac{3}{2}e^2a^2$, is a constant. Hence we conclude that (A.8) describes exactly the same field as that given by (2.2). Since the gauge term is absent in (2.2) we have subsequently, in equation (2.12), utilised a different transformation to (A.7) in order to transform the 2-surface $\sigma = $ constant, $r = $ constant into the form (2.6).

Appendix B

The linearised static axially symmetric Weyl form of the line element may be written

$$\Phi=(\theta^1)^2+(\theta^2)^2+(\theta^3)^2-(\theta^4)^2 \tag{B.1}$$

with

$$\begin{aligned}\theta^1&=(1+\nu-\lambda)\,d\rho\\ \theta^2&=(1+\nu-\lambda)\,dz\\ \theta^3&=\rho(1-\lambda)\,d\phi\\ \theta^4&=(1+\lambda)\,dt\end{aligned} \tag{B.2}$$

where $\lambda(\rho,z)=O_1$, $\nu(\rho,z)=O_1$. These 1-forms provide the components of an orthonormal tetrad. The nonvanishing components of the linearised Weyl tensor on this tetrad are

$$\begin{aligned}C_{1212}&=-C_{3434}=\tfrac{1}{3}(\Delta\lambda-\Delta\nu-2\rho^{-1}\lambda_1)\\ C_{3113}&=-\tfrac{1}{2}\rho^{-1}\nu_1-\tfrac{1}{6}\Delta\lambda-\lambda_{11}+\tfrac{2}{3}\Delta\lambda-\tfrac{1}{3}\rho^{-1}\lambda_1\\ C_{3123}&=C_{4124}=-\lambda_{12}-\tfrac{1}{2}\rho^{-1}\nu_2\\ C_{3223}&=-\lambda_{22}+\tfrac{1}{2}\rho^{-1}\nu_1-\tfrac{1}{6}\Delta\nu+\tfrac{2}{3}\Delta\lambda-\tfrac{1}{3}\rho^{-1}\lambda_1\\ C_{4114}&=-\lambda_{11}-\tfrac{1}{2}\rho^{-1}\nu_1+\tfrac{1}{6}\Delta\nu+\tfrac{1}{3}\Delta\lambda+\tfrac{1}{3}\rho^{-1}\lambda_1\\ C_{4224}&=-\lambda_{22}+\tfrac{1}{2}\rho^{-1}\nu_1+\tfrac{1}{6}\Delta\nu+\tfrac{1}{3}\Delta\lambda+\tfrac{1}{3}\rho^{-1}\lambda_1\end{aligned} \tag{B.3}$$

where $f_1=\partial f/\partial\rho$, $f_2=\partial f/\partial z$ and $\Delta f=\partial^2 f/\partial\rho^2+\partial^2 f/\partial z^2$ for $f=f(\rho,z)$.

The nonvanishing tetrad components of the Weyl tensor for the external field given by equations (4.13) are thus

$$2C^{\text{ext}}_{1212}=-C^{\text{ext}}_{3434}=C^{\text{ext}}_{3113}=-2C^{\text{ext}}_{3223}=2C^{\text{ext}}_{4114}=-C^{\text{ext}}_{4224}=-E^2. \tag{B.4}$$

Therefore C^{ext}_{abcd} is free from singularity, as one would expect.

We shall display here the nonvanishing tetrad components of the interaction-linearised Weyl tensor, obtained using equation (4.21) with all constants put to zero except e, E and b_2 in order to exhibit the fact that this field is free from wire singularities at $\theta=0$ or π. We carry out the calculation using (B.3) with the change of variable (4.17). We find

$$\begin{aligned}C^{\text{int}}_{1212}&=-C^{\text{int}}_{3434}=-b_2/r^3-(eE\cos\theta)/r^2+O_2\\ C^{\text{int}}_{3131}&=(b_2/r^3)(2-3\sin^2\theta)+(eE\cos\theta/r^2)(2-3\sin^2\theta)+O_2\\ C^{\text{int}}_{3123}&=C^{\text{int}}_{4124}=3b_2\sin\theta\cos\theta/r^3+3eE\sin\theta\cos^2\theta/r+O_2\\ C^{\text{int}}_{3232}&=(b_2/r^3)(2-3\cos^2\theta)-(eE\cos\theta/r^2)(1-3\sin^2\theta)+O_2\\ C^{\text{int}}_{4141}&=(b_2/r^3)(1-3\sin^2\theta)+(eE\cos\theta/r^2)(1-3\sin^2\theta)+O_2\\ C^{\text{int}}_{4242}&=(b_2/r^3)(1-3\cos^2\theta)-(eE\cos\theta/r^2)(2-3\sin^2\theta)+O_2.\end{aligned} \tag{B.5}$$

Putting $E=0$ we are left with the terms involving the constant b_2. These correspond, by equation (4.21), to

$$\hat{\lambda}=-b_2r^{-1},\qquad \hat{\nu}=0. \tag{B.6}$$

This is a disguised form of the linearised Schwarzschild solution with mass b_2. This is most easily seen by first writing out the linearised Weyl line element using equation (B.6), i.e.

$$ds^2 = (1+2b_2/r)(d\rho^2 + dz^2 + \rho^2\, d\phi^2) - (1-2b_2/r)\, dt^2 + O_2. \tag{B.7}$$

Now ρ and z are related to r and θ by equation (4.17). Up to now we always applied equation (4.17) to O_1 terms, and thus the O_1 term added to (4.17) would be automatically swallowed up in the O_2 error. This is not the case if we apply equation (4.17) to equation (B.7). The suitably modified transformation reads

$$\rho = (r - b_2)\sin\theta, \qquad z = a^{-1} + (r - b_2)\cos\theta. \tag{B.8}$$

When this is applied to equation (B7) we obtain

$$ds^2 = (1+2b_2/r)\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - (1-2b_2/r)\, dt^2 + O_2 \tag{B.9}$$

which is the linearised Schwarzschild solution, with the small mass parameter b_2, in a more familiar form. Since we do not want such an auxiliary mass in the external field we put $b_2 = 0$. Hence the tetrad components of the interaction-linearised Weyl tensor are in fact given by equation (B.5) with $b_2 = 0$.

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